

Notes on Matrix and Micro Strings*

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We review some recent developments in the study of M-theory compactifications via Matrix theory. In particular we highlight the appearance of IIA strings and their interactions, and explain the unifying role of the M-theory five-brane for describing the spectrum of the T^5 compactification and its duality symmetries. The 5+1-dimensional micro-string theory that lives on the fivebrane world-volume takes a central place in this presentation.

1. Introduction

In spite of considerable recent progress, M-theory is as yet a theory without a fundamental formulation. It is usually specified by means of a number of fundamental properties that we know it should possess. Among the most important of these characteristics is that the degrees of freedom and interactions of M-theory should provide a (mathematically consistent) representation of the maximally extended supersymmetry algebra of eleven dimensional supergravity [1]

$$\begin{aligned} \{Q_\alpha, Q_\beta\} = & (\Gamma^m)_{\alpha\beta} P_m + (\Gamma^{mn})_{\alpha\beta} Z_{mn}^{(2)} \\ & + (\Gamma^{m_1 \dots m_5})_{\alpha\beta} Z_{m_1 \dots m_5}^{(5)}. \end{aligned} \quad (1)$$

Here P_m denotes the eleven dimensional momentum, and $Z^{(2)}$ and $Z^{(5)}$ represent the two- and five-index central charges corresponding to the two types of extended objects present in M-theory, respectively the membrane and the five-brane. Upon compactification, these extended objects give rise to a rich spectrum of particles,

with many quantized charges corresponding to the various possible wrapping numbers and internal Kaluza-Klein momenta.

An alternative but presumably equivalent characterization of M-theory is that via compactification on a circle S^1 it becomes equivalent to ten dimensional type IIA string theory [2]. The particles with non-zero KK-momentum along the S^1 are in this correspondence identified with the D-particles of the IIA model, while the string coupling constant g_s emerges as the radius R of the S^1 via

$$\ell_s g_s = R, \quad \ell_s^2 = \ell_p^3 / R. \quad (2)$$

Here ℓ_s denotes the string scale and ℓ_p is the 11-dimensional Planck scale. Also upon further compactification, all M-theoretic charges now have well understood interpretations in terms of perturbative and non-perturbative string theoretic configurations, such as wrapped strings, D-branes [3] or solitonic five-branes [4]. The precise and well-defined perturbative rules for determining the interactions among strings and D-branes [3] [5] [6] [7] so far provides the best basis for concrete quantitative studies of M-theory dynamics. String theory still forms the solid foundation on which the new structure of M-theory needs to be built.

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From both the above starting points, a convincing amount of evidence has been collected to support the conjecture that toroidal M-theory compactifications exhibit a large group of discrete duality symmetries, that interchange the various internal charges [8]. While some of these dualities are most manifest from the eleven dimensional perspective, others become visible only in the string formulation. In combination, these so-called U-duality symmetries put very strong restrictions on the possible non-perturbative properties of M-theory, and in this way provide a wealth of information about its dynamics.

A very fruitful approach towards unraveling the mysteries of M-theory is provided by the matrix formulation initiated in [9], known as ‘Matrix theory.’ Its basic postulate is that the full eleven-dimensional dynamics of M-theory can be captured by means of the many-body quantum mechanics of N D-particles of IIA string theory, in the limit where N is taken to infinity. From the eleven dimensional perspective, this limit can be interpreted as applying a Lorentz boost, so that all degrees of freedom end up with a very large momentum along the extra eleventh direction.

One can formally combine this limiting procedure with a decompactification limit of the eleventh direction, so that the IIA string and M-theory compactification manifolds become identified. In this way, a fundamentally new perspective on non-perturbative string theory arises, in which its complete particle spectrum in a given dimension becomes identified with a relatively convenient subset of states compactified to one dimension less. Though in essence a light-cone gauge description, Matrix theory thus manages to provide a rather detailed theoretical framework for exploring the microscopic dynamics of M-theory, while preserving many of the geometrical foundations of string theory. There is now indeed a rather well-developed understanding of how Matrix theory incorporates all perturbative degrees of freedom and interactions of type II strings [10] [11] [12].

In these notes we will review some of the recent developments in this approach to M-theory. In particular we will discuss in detail the recently proposed formulation of M-theory com-

pactified on T^5 , based on the 5+1-dimensional world-volume theory of N NS 5-branes [14]. We will pay special attention to the appearance of the M5-brane in this proposal, and with the aim of illuminating the direct correspondence between the framework of [14] and our earlier studies [4]. In addition we will highlight the main ingredients of Matrix string theory [12], and discuss its application to the study of the interacting string-like theory on the 5-brane worldvolume.

2. U-duality and the M5-brane

The degrees of freedom and interactions of M-theory compactified on a five-torus T^5 provide a representation of the maximal central extension of the 5+1 dimensional $\mathcal{N} = 4$ supersymmetry algebra, which in a light-cone frame takes the form²

$$\begin{aligned} \{\tilde{Q}_\alpha^a, \tilde{Q}_\beta^b\} &= p_+ \delta_{\alpha\beta} \mathbf{1}^{ab} \\ \{Q_\alpha^a, \tilde{Q}_\beta^b\} &= p_I \gamma_{\alpha\beta}^I \mathbf{1}^{ab} + \delta_{\alpha\beta} Z^{ab} \\ \{Q_\alpha^a, Q_\beta^b\} &= p_- \delta_{\alpha\beta} \mathbf{1}^{ab} \end{aligned} \quad (3)$$

Here the internal labels $a, b = 1, \dots, 4$ are $SO(5)$ spinor indices, and $I = 1, \dots, 4$ label the four transverse uncompactified space dimensions. The central charge matrix Z^{ab} comprises a total of 16 internal charges, which split up in five quantized Kaluza-Klein momenta p_i with $i = 1, \dots, 5$, ten charges $m_{ij} = -m_{ji}$ that specify the winding numbers of the M membranes around the five-torus, and one single charge q_5 that represents the wrapping number of the M5 brane. To make this interpretation more manifest, it is often convenient to expand Z_{ab} in terms of $SO(5)$ gamma matrices as

$$Z_{ab} = q_5 \mathbf{1}_{ab} + p_i \gamma_{ab}^i + m_{ij} \gamma_{ab}^{ij}. \quad (4)$$

More generally, the decomposition of Z_{ab} into integral charges depends on the moduli of the T^5 compactification, which parametrize the coset manifold

$$\mathcal{M} = SO(5, 5)/SO(5) \times SO(5). \quad (5)$$

²Hence α and β are spinor indices of the transversal $SO(4)$ rotation group.

The U-duality group is the subgroup of $SO(5, 5)$ rotations that map the lattice of integral charges on to itself, and is thus identified with

$$\mathcal{U} = SO(5, 5, \mathbf{Z}). \quad (6)$$

From the eleven dimensional viewpoint, the moduli space (5) is parametrized by the metric G_{mn} and three-form potential C_{mnk} defined on the internal manifold T^5 . After replacing the 3-form C on T^5 by its Hodge-dual 2-form $B = *C$, this parametrization of \mathcal{M} reduces to the familiar one from the toroidal compactification of type II string theory on T^5 . This correspondence, which is strengthened by the fact that the U-duality group (6) takes the form of the T^5 T-duality group, has become a central theme in recent attempts to formulate the T^5 compactification of M-theory [4,14].

To study the spectrum of M-theory on T^5 , it is natural to start from the maximal dimensional object that can wrap around the internal manifold, which is the M5 brane. From its description as a soliton it is known that its low energy collective modes form a tensor multiplet of the $\mathcal{N} = (2, 0)$ world-brane supersymmetry, consisting of five scalars, an anti-symmetric tensor with self-dual field strength $T = dU$ and 4 chiral fermions. As shown in [4], this worldvolume field theory can be extended to include all the above 16 charges of M-theory on T^5 via an appropriate identification of flux configurations.

To make this concrete, consider a M5-brane with topology of $T^5 \times \mathbf{R}$ where \mathbf{R} represents the world-brane time τ . To begin with, we can include configurations carrying a non-zero flux

$$m_{ij} = \int_{T^3_{ij}} dU \quad (7)$$

through the 10 independent three-cycles T^3_{ij} in T^5 . An M5-brane for which these charges are non-zero represents a bound state with a corresponding number of membranes wrapped around the dual 2-cycle. The charges m_{ij} are part of a 16-dimensional spinor representation of the U-duality group $SO(5, 5, \mathbf{Z})$, which is isomorphic to the odd (co)homology of the five-torus T^5 . This observation motivates us to try to write the other

charges q_5 and p_i as fluxes of 5- and 1-form field strengths through the 5-cycle and 1-cycles on T^5 .

Before presenting this construction, let us mention that in course we will in fact arrive at a rather new notion of the M5-brane worldvolume theory, that is rather different from the usual one in terms of the physical location of the fivebrane soliton in space-time. In particular, the M5-brane worldvolume theory we will arrive at lives on an auxiliary 5+1-dimensional parameter space, denoted by \tilde{T}^5 . This auxiliary space is only indirectly related to the embedded M5-brane soliton in physical space-time, via an appropriate geometrical interpretation of the worldvolume fields.

In this new set-up, we can introduce a winding number of the M5-brane via

$$q_5 = \int_{\tilde{T}^5} dV, \quad (8)$$

where the 5-form dV is interpreted as the pull-back to the auxiliary torus \tilde{T}^5 of the constant volume element on the target T^5 . We would like to turn the 4-form potential V into an independent field, that is part of the world-brane theory.

To this end we imagine that the $(2, 0)$ world-brane tensor multiplet in fact provides an effective *light-cone description* of the M5-brane. Concretely this means that of the five scalars, we will interpret only four as describing transversal coordinates in space-time. This leaves us with one additional scalar field Y . Since the world-brane is 5+1-dimensional, this scalar can be dualized to a 4-form, which we will identify with the 4-form V used in (8). Hence we will assume the 5-form field strength $W = dV$ associated with V is normalized such that the associated flux is a non-negative integer.

Quite generally, in a light-cone formalism for extended objects one typically obtains a residual gauge symmetry under volume preserving diffeomorphisms [15]. For the case of the 5-brane these can be used to eliminate all dependence on the five compact embedding coordinates X^i (that map the auxiliary \tilde{T}^5 into the target T^5), except for the volume-form

$$dV = dX^i \wedge dX^j \wedge dX^k \wedge dX^l \wedge dX^m \epsilon_{ijklm}. \quad (9)$$

The five spatial components of $*V^i$ can roughly be identified with the embedding coordinates X^i . In this correspondence the linearized gauge-transformation $V \rightarrow V + d\Lambda$ of the 4-form potential describes a volume preserving diffeomorphism, since it leaves dV invariant.

We can formalize the duality transformation between V and Y by taking W to be an independent 5-form and introducing Y as the Lagrange multiplier that imposes the Bianchi identity $dW = 0$. The fact that W has integral fluxes implies that Y must be a periodic field, *i.e.* $Y \equiv Y + 2\pi r$ with r integer. The 5 remaining charges p_i can thus be identified with the integer winding numbers of Y around the 5 one-cycles

$$p_i = \oint_{T_i^1} dY. \quad (10)$$

Since on-shell $\Pi_V \equiv \delta S / \delta \dot{V} = dY$, we deduce that dY is the canonical momentum for V . From this it is straightforward to show that these operators are indeed the generators of translations along the internal directions on the target space 5-torus. The formula for the M5-brane wrapping number now becomes

$$q_5 = \int_{\tilde{T}^5} \Pi_Y, \quad (11)$$

with Π_Y the conjugate momentum to Y .

In this way all the 16 charges (q, p_i, m_{ij}) have been written as fluxes through the odd homology cycles on the M5-brane. These 16 bosonic zero modes form the world-volume superpartners of the 16 fermionic zero modes, that represent the space-time supersymmetries that are broken by the M5-brane soliton. Based on this result, it was shown in [4] that the (2,0) worldvolume theory of the M5-brane could be used to represent the full maximally extended space-time supersymmetry algebra (3).

The distinction between the above and the standard notion of the 5-brane world-volume is perhaps best explained in analogy with the worldsheet description of toroidally compactified string theory. According to the strict definition, the worldsheet of a string refers to its particular location in space-time, and thus also specifies a given winding sector. For strings, however, it is well

known to be advantageous to combine into the worldsheet CFT all superselection sectors corresponding to all possible momenta and winding numbers. T-duality for example appears as a true symmetry of the worldsheet CFT only if it includes all sectors. The notion of the M5 world-brane theory used here is indeed the direct analogue of this: just as for strings, the various wrapping numbers and momenta appear as quantized zero modes of the worldvolume fields. A second important analogy is that, just like the CFT Hilbert space includes unwrapped string states, we can in the above set-up also allow for unwrapped M5 brane states with $q_5 = 0$.

This naturally leads to the further suggestion that it should be possible to extend the M5 worldbrane theory in such a way that U-duality becomes a true symmetry (just like T-duality in toroidal CFT). In [4] it was shown that this can indeed be achieved by assuming that the worldvolume theory on \tilde{T}^5 is described by a 5+1-dimensional second quantized string theory, whose low energy effective modes describe the (2,0) tensor multiplet. Via the $SO(5,5, \mathbf{Z})$ T-duality of this “micro-string theory”, this produces a manifestly U-duality invariant description of the BPS states of all M-theory compactification down to 6 or more dimensions [4].

Until quite recently, only limited theoretical tools were available to analyze the detailed properties of this conjectured micro-string theory, or even to confirm its existence. The traditional methods usually start from the physical world volume of a macroscopic M5-brane soliton, and the micro-string theory is in fact not easily visible from this perspective. As seen e.g. from the identifications (10) and (11), the local physics on the target space torus T^5 is somewhat indirectly related to that on the parameter space \tilde{T}^5 where the micro-string lives. The specification of a particular winding sector q_5 , for instance, breaks the U-duality group to geometric symmetry group $SL(5, \mathbf{Z})$, and thus hides the stringy part of the micro-string T-duality symmetry. This illustrates that it may be misleading to think about the micro-strings directly as physical strings moving on the space-time location of the M5-brane soliton.

3. BPS Spectrum

Further useful information about the relation between the auxiliary and target space can be obtained by considering the masses of the various BPS states.

The world-brane theory carries a chiral $\mathcal{N}=2$ supersymmetry algebra. To write this algebra, it is convenient to introduce chiral $SO(4)$ spinor indices α and $\dot{\alpha}$ for the transverse rotation group (which represents an R-symmetry of the world-volume supersymmetry). We then have

$$\begin{aligned}\{Q_a^\alpha, Q_b^\beta\} &= \epsilon^{\alpha\beta}(\gamma_{ab}^0 H + \gamma_{ab}^i(P_i + W_i)), \\ \{Q_a^{\dot{\alpha}}, Q_b^{\dot{\beta}}\} &= \epsilon^{\dot{\alpha}\dot{\beta}}(\gamma_{ab}^0 H + \gamma_{ab}^i(P_i - W_i)).\end{aligned}\quad (12)$$

Here H is the world-volume Hamiltonian, and P_i is the integrated momentum flux through the five-torus

$$H = \int_{\tilde{T}^5} T_{00}, \quad P_i = \int_{\tilde{T}^5} T_{0i}, \quad (13)$$

where T_{mn} denotes the world-volume energy-momentum tensor. Hence H and P_i represent the generators of time and space translation on the 5+1-dimensional parameter space-time. The operators W_i that appear in the algebra (12) represent a vector-like central charge, and is a reflection of the (possible) presence of string-like objects in the world-volume theory. This vector charge W_i indeed naturally combines with the momenta P_i into a 10 component vector representation of the $SO(5, 5, \mathbf{Z})$ world-volume T-duality group.

The world-brane supercharges represent the unbroken part of space-time supersymmetry algebra. The other generators must be identified with fermionic zero-modes, that are the world-brane superpartners of the bosonic fluxes described above. In order to exactly reproduce the 5+1-dimensional space-time supersymmetry (3), however, we must project onto states for which the total momentum fluxes P_i and vector central charges W_i all vanish

$$P_i = 0, \quad W_i = 0. \quad (14)$$

These are the analogues of the usual level match-

ing relations of light-cone string theory. In addition, we must impose

$$H = p_- \quad (15)$$

which reflects the identification of the world-volume time coordinate τ with the space-time light-cone coordinate x^+ . The mass-shell condition $p_+ p_- = m^2$ (we assume for simplicity that all transverse space-time momenta p_I are zero) relates the space-time mass of M5-brane configuration to the world-volume energy H via

$$m^2 = p_+ H \quad (16)$$

This relation, together with identification of the central charge Z_{ab} as world-brane fluxes, uniquely specifies the geometric correspondence between the auxiliary torus \tilde{T}^5 and the target space torus T^5 . To make this translation explicit, let us denote the lengths of the five sides of each torus by \tilde{L}_i and L_i respectively, and let us fix the normalization of the fields dU and dY in accordance with the flux quantization formulas (7), (10), and (11). Since the left-hand sides are all integers, this in particular fixes the dimensionalities of the two-form U and scalar Y .

To simplify the following formulas, we will further use *natural units* both in the M-theory target space, as well as on the auxiliary space. In the target space, these natural units are set by the eleven dimensional Planck scale ℓ_p , while on the world-volume, the natural units are provided by the micro-string length scale $\tilde{\ell}_s$. Apart from the size of the five torus \tilde{T}^5 , this string length $\tilde{\ell}_s$ is indeed the only other length scale present in the 5+1-dimensional world-volume theory.

Now let us consider the physical masses of the various 1/2 BPS states of M-theory. These states can carry either pure KK momentum charge, or pure (non-intersecting) membrane and fivebrane wrapping number. From the target-space perspective, we deduce that these states have the following masses (in 11-dimensional Planck units $\ell_p = 1$)

$$\begin{aligned}\text{KK momenta:} \quad & m^2 = (p_i/L_i)^2 \\ \text{membranes:} \quad & m^2 = (m_{ij} L_i L_j)^2 \\ \text{fivebranes:} \quad & m^2 = (q_5 V_5)^2\end{aligned}\quad (17)$$

where $V_5 = L_1 \dots L_5$ denotes the volume of the M-theory five torus T^5 in Planckian units.

In the world-volume language these 1/2 BPS states all correspond to the supersymmetric ground states that carry a single flux quantum number. Their mass is therefore determined via the mass-shell relation (16) by the world-volume energy carried by the corresponding flux configuration. To deduce this energy, it is sufficient to note that at very long distances, the world-volume theory reduces to a non-interacting free field theory of the (2,0) tensor multiplet. Hence the flux contribution to the M5-brane Hamiltonian can be extracted from the free field expression³

$$H = \frac{1}{\beta} \int_{\tilde{T}^5} d^5x \left[|dY|^2 + |dU|^2 + |\Pi_Y|^2 \right] \quad (18)$$

where β is an arbitrary parameter (to be related later on to the light-cone momentum p_+ in space-time). Now using (7), (10), and (11), we deduce that the 1/2 BPS states have the following world-volume energies (in micro-string units $\tilde{\ell}_s = 1$)

$$\begin{aligned} \text{KK momenta:} \quad & \frac{\tilde{V}_5}{\beta} \left(\frac{p_i}{\tilde{L}_i} \right)^2 \\ \text{membranes:} \quad & H = \frac{(m_{ij} \tilde{L}_i \tilde{L}_j)^2}{\beta \tilde{V}_5} \\ \text{fivebranes:} \quad & H = \frac{q_5^2}{\beta \tilde{V}_5} \end{aligned} \quad (19)$$

Here $\tilde{V}_5 = \tilde{L}_1 \dots \tilde{L}_5$ denotes the volume of the auxiliary space \tilde{T}^5 in string units.

From the condition that all these three expressions reproduce the correct mass formulas (17) via (16), we can now read off the M-theory parameters in terms of the world-volume parameters. First of all we see that the two five tori T^5 and \tilde{T}^5 in fact have the same *shape*, *i.e.* they are related by means of an overall rescaling

$$L_i = v \tilde{L}_i \quad (20)$$

with v some dimensionless number. The remaining M-theory parameters we need to determine

³Here we have used that the self-duality relation $\Pi_U = dU$. We further left out the transverse scalars X_I since their fluxes are set to zero.

are V_5 and p_+ , and we need to express them in terms of \tilde{V}_5 and β in (18). A little algebra show that

$$V_5 = \frac{1}{\tilde{V}_5^{2/3}}, \quad p_+ = \frac{\beta}{\tilde{V}_5^{1/3}}. \quad (21)$$

In particular we find that the wordvolume radii are related to the M-theory radii by

$$\tilde{L}_i = \frac{L_i}{V_5^{1/2}}. \quad (22)$$

Hence we see in particular that the decompactification limit of M-theory corresponds to the zero volume limit of the auxiliary torus.

Of course, here we have to take into account that T-duality of the micro-string theory can relate different limits of the \tilde{T}^5 torus geometry. This fact plays an important role when we consider partial decompactifications. For example, if we send one of the radii of the M-theory five-torus, say L_5 , to infinity, then (22) tells us that, after a T-duality on the remaining four compact directions, the worldvolume of \tilde{T}^5 tends to infinity. In this way we deduce that the BPS states of the T^4 compactification of M-theory is described by the zero slope limit of the micro-string theory on a T^5 world-volume. This is of importance in understanding the $SL(5, \mathbf{Z})$ U-duality group of M-theory compactifications on T^4 [16,17]. Indeed, all U-duality symmetries for T^d compactifications with $d \leq 5$ can be understood in this way as geometric symmetries of the micro-string model.

Finally, let us mention that as soon as one considers states with more general fluxes, they are at most 1/4 BPS and the mass formula becomes more involved. One finds

$$m_{\text{BPS}}^2 = m_0^2 + 2V_5 \sqrt{(\mathcal{K}_i/L_i)^2 + \frac{1}{V_5^2} (\mathcal{W}_i L_i)^2}, \quad (23)$$

with

$$\begin{aligned} m_0^2 &= (q_5 V_5)^2 + (p_i/L_i)^2 + (m_{ij} L_i L_j)^2, \\ \mathcal{K}_i &= q_5 p_i + \frac{1}{2} \epsilon_{ijklm} m_{jk} m_{lm}, \end{aligned} \quad (24)$$

$$\mathcal{W}_i = m_{ij} p_j.$$

These formulas are completely duality invariant, and were reproduced via a world-volume analysis

in [4]. A crucial new ingredient in the description of BPS states with a non-zero value for \mathcal{K}_i is that they necessarily contain worldbrane excitations with non-zero momentum on \tilde{T}^5 ,⁴ while $\mathcal{W}_i \neq 0$ necessarily describes states with non-zero winding number for the micro-strings. Hence, without the micro-string degrees of freedom, the M5-brane would have had no BPS-states with non-zero value for \mathcal{W}_i . The geometrical interpretation of the bilinear expressions for \mathcal{K}_i and \mathcal{W}_i will become more clear in the following sections.

4. M-theory from NS 5-branes

The above attempt to unify the complete particle spectrum of a given M-theory compactification in terms of one single world-volume theory has now found a natural place and a far reaching generalization in the form of the matrix formulation of M-theory initiated in [9]. While the previous sections were in part based on an inspired guess, motivated by U-duality and the form of the space-time supersymmetry algebra, in the following sections we will describe how the exact same structure can be derived from the recently proposed Matrix description [14] of M-theory on T^5 . So we will again start from the beginning.

According to the Matrix theory conjecture, one can represent all degrees of freedom of M-theory via an appropriate large N limit of the matrix quantum mechanics of the D0-branes [9]. The true extent this proposal becomes most visible when one combines this large N limit with a decompactification limit $R \rightarrow \infty$ of the 11-th M-theory direction. The ratio

$$p_+ = N/R \quad (25)$$

represents the light-cone momentum in the new decompactified direction.

U-duality invariance implies that any charge of compactified IIA string theory can in principle be taken to correspond to the eleventh KK momentum N . One can thus reformulate the original

⁴This is implied by the level-matching condition $P_i = 0$, since \mathcal{K}_i in fact represents the zero-mode contribution to P_i . Notice further that for $q_5 \neq 0$, \mathcal{K}_i contains a contribution proportional to q_5 times the space-time KK momentum p_i , which indeed reflects the geometric relation between world-volume and target space translations.

matrix conjecture of [9] by applying some duality symmetry of the type II string to map the D0-branes to some other type of branes. Compactification of matrix theory indeed generically proceeds by using the maximal T-duality to interchange the D0 brane charge by that of the maximal dimensional D p-brane that can wrap around the internal manifold. In our specific example of T^5 this would be the D5-brane. Alternatively, as was first suggested in [14] (following up earlier work [16] [17] [4]), one can choose to identify this charge N with the NS five-brane wrapping number around the five-torus T^5 .

In the following table we have indicated the chain of duality mappings that provides the dictionary between the original matrix set-up of [9] and the dual language of [14]:

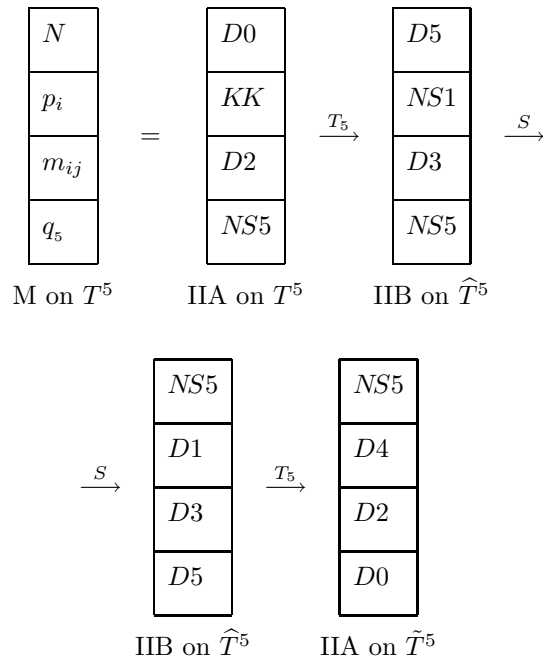


Table 1. Interpretation of the quantum numbers and their translation under the duality chain.

We notice that all 16 quantized charges, that make up the central charge matrix Z_{ab} , have become D-brane wrapping numbers in the last two columns, and furthermore that the $SO(5, 5, \mathbf{Z})$ U-duality symmetry is mapped onto the (manifest) T-duality symmetry of the type II theory.

Note further that the D-brane charge vector indeed transforms in a spinor representation of the T-duality group.

The string coupling, string scale and dimensions of the five torus are related through this duality chain as follows.⁵ The chain consists of a T_5 duality on all five direction of the T^5 , an S-duality of the IIB theory, and finally again a maximal T-duality on T^5 . The end result is a mapping from large N D-particle quantum dynamics to that of N NS five-branes in IIA theory.

On the left, we have started with the conventional definition of M-theory compactified on a five-torus T^5 , with dimensions L_i , $i = 1, \dots, 5$, as the strong coupling limit of type IIA string theory on the same torus T^5 . Under the first T_5 duality, a D0-brane becomes the D5-brane wrapped around the dual torus \hat{T}_5 with dimensions

$$\Sigma_i = \frac{\ell_s^2}{L_i} = \frac{1}{RL_i}. \quad (26)$$

Here we used (2), and again work in eleven dimensional Planck units $\ell_p = 1$. Following the usual rule of T -duality, the string coupling g_s gets a factor of the volume in string units, so the dual coupling \hat{g}_s becomes (see again (2))

$$\hat{g}_s = \frac{g_s \ell_s^5}{V_5} = \frac{1}{RV_5} \quad (27)$$

where V_5 is again the five-volume of T^5 (in Planck units). The subsequent S-duality transformation maps the D5-brane onto the IIB NS5-brane, inverts the string coupling, and changes the string scale by a factor of $\sqrt{\hat{g}_s}$. So we have

$$\tilde{g}'_s = V_5 R, \quad \tilde{\ell}_s^2 = \frac{1}{R^2 V_5}. \quad (28)$$

The size of the five torus of course remains unchanged under the S-duality.

Finally we perform the T_5 duality again, so that N counts IIA fivebrane charge. The dimensions of the resulting five-torus \tilde{T}^5 are

$$\tilde{L}_i = \frac{L_i}{RV_5}. \quad (29)$$

⁵For this section we benefited from valuable discussions with G. Moore. A similar presentation can indeed be found in [27].

while the string coupling \tilde{g}_s of the IIA string after the $T_5 ST_5$ transformation becomes

$$\tilde{g}_s^2 = \frac{R^2}{V_5} \quad (30)$$

The string scale of the IIA theory is the same as in (28).

In [14] convincing arguments were given for the existence of a well-defined limit of type II string theory, that isolates the world volume dynamics of N NS five-branes from the 9+1-dimensional bulk degrees of freedom. The key observation made in [14] is that, even in the decoupling limit $\tilde{g}_s \rightarrow 0$ for the bulk string interactions, one is still left with a non-trivial interacting 5+1 dimensional theory that survives on the world volume of the 5-branes. This theory depends on only two dimensionless parameters, namely N and the size of the torus \tilde{T}^5 in units of the string length $\tilde{\ell}_s$. If one keeps the last ratio finite, the system will contain string like degrees of freedom. According to the proposal put forward in [14], this world volume theory of N NS 5-branes in the limit of large N may provide a complete matrix theoretic description of M-theory compactified on T^5 .

This proposal in fact needs some explanation, since as seen from (30) this decoupling limit $\tilde{g}_s \rightarrow 0$ in fact contradicts the decompactification limit $R \rightarrow \infty$ that is needed for obtaining M-theory on T^5 . The resolution of this apparent contradiction should come from the particular properties of the world-volume theory of N 5-branes for large N . As we will see later, via a mechanism familiar e.g. from studies of black holes in string theory [24] or matrix string theory [10][11][12], the system of N wrapped NS 5-branes contains a low energy sector with energies of order $1/N$ times smaller than the inverse size of the five-torus \tilde{T}^5 . These low energy modes are the ones that will become the M-theory degrees of freedom, and the large N limit must be taken in such a way that only these states survive. This implies that the typical length scale of the relevant configurations is larger by a factor of N than the size of \tilde{T}^5 .

When one analyzes this low lying energy spectrum for very large N , one discovers that it becomes essentially independent of the string coupling constant \tilde{g}_s , provided \tilde{g}_s does not grow too

fast with N . It thus seems reasonable to assume that in an appropriate large N limit, the world-volume theory of the NS 5-branes itself becomes independent of \tilde{g}_s . This opens up the possibility that the weak coupling limit that was used in [14] for proving the existence of the 5+1-dimensional worldbrane theory, in fact coincides with the strong coupling regime that corresponds with the decompactification limit of M-theory.

Before we continue, let us remark that this Matrix theoretic set-up can already be seen to reproduce many qualitative and quantitative features of the framework described in the previous sections. Evidently, the world-volume theory of the N NS 5-branes plays a very analogous role to the proposed M5-brane world-volume theory of sections 2 and 3. Most importantly, in both set-ups the U-duality symmetry of M-theory becomes identified with a T-duality of a 5+1-dimensional worldvolume string theory. In addition, as we will see a little later, all 16 charges of M-theory arise in the present Matrix theory set-up [14] via the exact same identification of fluxes as described in section 2.

Finally, by comparing the formulas (28) and (29) with those of the previous section, we see that the size and shapes of the respective auxiliary five-tori \hat{T}^5 and \tilde{T}^5 (notice however the subtle difference in notation) are expressed via an identical set of relations in terms of those the M-theory torus T^5 . The micro-string torus \tilde{T}^5 of sections 2 and 3 is indeed related the Matrix theory torus \hat{T}^5 used above via a rescaling with a factor N . Since (as will be elaborated later on) the micro-string scale $\tilde{\ell}_s$ is also N times larger than the fundamental string scale, the two auxiliary five tori have the same size relative to the respective string scales. This fact ensures that the standard T-duality symmetry of the small \tilde{T}^5 indeed translates into an isomorphic T-duality group that acts on the large \hat{T}^5 on which the micro-string lives.

Via this formulation [14], Matrix theory thus naturally incorporates our earlier ideas on the M5-brane. More importantly, however, it also provides a concrete prescription for including interactions, both among the M5-branes as well as for the micro-string theory inside its world-

volume.

5. SYM theory on T^5

One possible starting point for concretizing this matrix description of M-theory on T^5 is provided by the large N limit of 5+1-dimensional super Yang-Mills theory

$$S = \frac{1}{g^2} \int d^6x \operatorname{tr} \left(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (D_\mu X^I)^2 + \psi \Gamma^i D_i \psi + \psi \Gamma^I [X^I, \psi] + \frac{1}{4} [X^I, X^J]^2 \right)$$

Here $I = 1, \dots, 4$ labels the 4 Higgs scalar fields. This theory can be thought of as describing the low energy collective modes of N D5 branes of IIB string theory (the second column in the above table), wrapped around the dual five-torus \hat{T}^5 . In terms of the parameters above, the Yang-Mills coupling of the D5-brane world-volume theory is then

$$\frac{1}{g_{YM}^2} = \frac{1}{\tilde{g}_s \ell_s^2} = V_5 R^2 \quad (31)$$

Alternatively, we can apply an S-duality and view the large N SYM model as the low energy world volume theory of N IIB NS 5-branes. Note that the above YM coupling then becomes equal to the fundamental NS string tension $g_{YM} = \tilde{\ell}_s$.

According to the prescription of Matrix theory, finite energy states of M-theory on T^5 are obtained by restricting to states of the SYM model with energies of order $1/N$. Since these states are all supported at scales much larger than the string scale, it seems an allowed approximation to discard the stringy corrections to the $U(N)$ SYM description of the NS 5-brane dynamics. An obvious subtlety with making this low-energy truncation is that the 5+1 dimensional SYM model in itself does not represent a well-defined renormalizable field theory. It is conceivable, however, that via the combined large N and IR limit that is needed for Matrix theory, one can still extract well-defined dynamical rules for the subset of the SYM degrees of freedom that are relevant for the description of M-theory on T^5 .

A second remark is that the Yang-Mills description in fact seems incomplete. Namely, to describe all possible quantum numbers, one should

be able to include all possible bound states of the NS 5-branes with the various D-branes. Generally these are described by turning on appropriate fluxes of the SYM theory. For the D1-branes and D3-branes, the corresponding winding number are represented via the electric and magnetic flux quantum numbers

$$p_i = \int_{T_i^4} \text{tr } E, \quad m_{ij} = \int_{T_{ij}^2} \text{tr } F, \quad (32)$$

through the dual 4 and 2-cycles of the \widehat{T}^5 , respectively. The D5 brane wrapping number, however, does not seem to appear naturally as a flux of the SYM theory. This fact has so far been the main obstacle for incorporating the transverse M5 brane into Matrix theory.

Central in the correspondence between the large N SYM theory and M-theory on T^5 is the fact that (apart from this problem of the missing q_5 -charge) the maximally extended supersymmetry algebra (3) can be represented in terms of the SYM degrees of freedom [20]. The construction is quite standard from the general context of string solitons, and relies on the identification of the world-volume supersymmetry charges of the SYM model

$$Q = \int \text{tr} \left[\bar{\theta} \left(\gamma^i E_i + \gamma^I \Pi_I - \frac{1}{2} (\gamma^{ij} F_{ij} + \gamma^{IJ} D_i X_J + \gamma^{IJ} [X_I, X_J]) \right) \right] \quad (33)$$

with the generators the unbroken space-time supersymmetries, while the generators of the broken supertranslations are represented by the fermion zero modes

$$\tilde{Q} = \int \text{tr } \theta. \quad (34)$$

The dynamical supersymmetry charges (33) generate an algebra of the exact form (12), where

$$P_i = \int \text{tr} (E_i F_{ij} + \Pi^I D_i X + \theta^T D_i \theta) \quad (35)$$

is the momentum flux and where

$$W_i = \int \text{tr} (F \wedge F)_i \quad (36)$$

is the instanton winding number. Hence the SYM field theory explicitly reveals the presence of the string like degrees of freedom in the form of the non-abelian instanton configurations. As we will describe in a later section, the long distance dynamics of these instanton strings may be used as a quantitative description of the interacting micro-string theory.

If k_i denotes the $SU(N)$ Yang-Mills instanton number, we evaluate

$$NW_i = \frac{1}{2} (m \wedge m)_i + N k_i. \quad (37)$$

As we will make more explicit in a moment, the right-hand side can indeed be seen to correspond to a total string winding number.⁶ To see this (and to get more insight into how one might recover the M5-brane from the present set-up) it will be useful to consider the world-volume properties of the IIA and IIB NS 5-branes from a slightly different perspective.

6. Winding Microstrings

After the ground breaking work [22] on the entropy of 5+1-dimensional black holes, it became clear that the internal dynamics of the string theory 5-branes are most effectively described in terms of an effective world-volume string theory [4,24]. These string-like degrees of freedom, which in the following we will also call micro-strings, arise in part due to one-dimensional intersections [22] that occur between D-branes, but also due to fundamental NS strings trapped inside NS 5-branes [21][14]. As indicated in *fig. 1*, the relation between the world-volume string theory and the low-energy collective modes of the 5-branes indeed parallels that between the fundamental type II strings and the low energy effective field theory in space-time. This correspondence predicts in particular the precise form of the micro-string ground states.

In the past year, this effective string theory has been developed into a remarkably successful

⁶This is true provided the $U(N)$ symmetry is essentially unbroken by the Higgs scalars. More generally, if some of the Higgs scalars have expectation values that are substantially different from zero, W_i will naturally split up into separate contributions corresponding to instantons of the smaller unbroken subgroups.

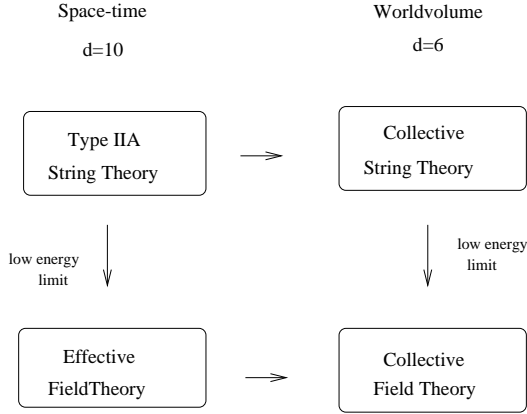


Fig. 1: *The effective world-volume dynamics of the solitonic NS 5-brane defined at $d = 6$ “collective string theory”, whose low-energy degrees of freedom coincide with the usual collective fields.*

framework for describing the statistical and dynamical properties (such as absorption and emission processes) of the 5-brane black holes. Until recently, however, it was unclear to what extent the corresponding 5+1-dimensional string could really be isolated as a separate theory, that exists independently from the 10-dimensional type II string. The elegant argumentation of [14] provides convincing evidence that this is indeed the case. We will now collect some useful topological properties of the micro-string theory.

A D-brane in type IIB string theory has odd dimension, and thus a general configuration of D-branes wound around the internal manifold T^5 will be characterized by a 16 component vector $\mathbf{q}^B \in H_{\text{odd}}(T^5, \mathbf{Z})$ in the odd integral homology of T^5 . Hence the corresponding D-brane intersection strings carry a total winding number given by the intersection form on $H_{\text{odd}}(T^5, \mathbf{Z})$, which in terms of our labeling of charges (see Table 1 in section 3) takes the form

$$\frac{1}{2}(\mathbf{q}^B \wedge \mathbf{q}^B)_i = q_5 p_i + \frac{1}{2}(m \wedge m)_i \quad (38)$$

Fundamental NS strings can also manifest themselves as string-like objects within the 5-brane world volume. Inside a bound state with N NS 5-branes, each NS string can in fact break up into

N separate strings with reduced string tension $1/N\alpha'$. These fractional strings are also intersection strings, and string duality indeed implies that they are indistinguishable (in the sense of statistics) from the above D-brane intersection strings. Denoting the NS string winding charge by k_i , the total number of intersection strings is given by⁷

$$NW_i^B = \frac{1}{2}(m \wedge m)_i + q_5 p_i + Nk_i \quad (39)$$

This formula generalizes the formula (37) for the instanton winding charge to the case where $q_5 \neq 0$. Notice further the correspondence with equation (24).

Let us now switch to the IIA perspective. In this case D-branes have even dimensions and thus D-brane configurations are characterized by $\mathbf{q}^A \in H_{\text{even}}(T^5, \mathbf{Z})$, and string-like intersections occur between 2- and 4-branes. Thus their total winding number is determined via

$$\frac{1}{2}(\mathbf{q}^A \wedge \mathbf{q}^A)_i = m_{ij} p_j \quad (40)$$

which is the intersection form on $H_{\text{even}}(T^5, \mathbf{Z})$. Hence, via the same mechanism as described for the IIB case, the total micro-string winding number (including the intersection strings between fundamental strings the N fivebranes) now takes the form

$$NW_i^A = m_{ij} p_j + Nk_i \quad (41)$$

where k_i denotes the IIA string winding number.

As the IIA theory and IIB model are related via T-duality, the two micro-string theories are actually the same, except that the total winding numbers W_i get interchanged with the total momentum fluxes P_i . From the above formulas it is easy to see that the flux contribution to e.g. the IIA string winding number W_i^A is indeed identical to the flux contribution to the IIB momentum flux P_i^B given in (35). A similar check can also be done for the IIB winding charge W_i^B and the dual IIA momentum flux P_i^A .

Below we have indicated the Matrix-theory interpretation the total momentum flux P_i and

⁷Here we assume that all strings and branes are at the same location in the non-compact space-time. See also previous footnote.

micro-string winding number W_i on the auxiliary T^5 , as obtained via the $T_5 S T_5$ duality chain:

$$\begin{array}{ccccc}
 \boxed{\begin{array}{c} P_i \\ W_i \end{array}} & = & \boxed{\begin{array}{c} \text{NS1} \\ D4 \end{array}} & \xrightarrow{T^5} & \boxed{\begin{array}{c} \text{KK} \\ D1 \end{array}} & \xrightarrow{S} \\
 \text{M on } T^5 & & \text{IIA on } T^5 & & \text{IIB on } \hat{T}^5 & \\
 & & & & & \\
 & & \xrightarrow{S} & \boxed{\begin{array}{c} \text{KK} \\ \text{NS1} \end{array}} & \xrightarrow{T^5} & \boxed{\begin{array}{c} \text{NS1} \\ \text{KK} \end{array}} \\
 & & & \text{IIB on } \hat{T}^5 & & \text{IIA on } \tilde{T}^5
 \end{array}$$

Hence non-zero values of the charges P_i and W_i in Matrix theory indicate the presence of longitudinally wrapped M2 and M5 branes. Finite energy M-theory configurations are therefore selected by imposing level matching constraints $P_i = 0$ and $W_i = 0$, which in the large N NS 5-brane system (the last two columns) project onto the sector with vanishing KK momenta and NS winding number. The same level matching were introduced already in section 3 from the condition that the space-time supersymmetry algebra (3) be reproduced.

7. The Matrix M5-brane

Let us now put the description of sections 2 and 3 of the M5-brane in a proper Matrix theoretic perspective. The correspondence will be most clear in the formulation of Matrix theory as the large N limit of N IIA 5-branes.

The fields parametrizing the low energy collective modes of a single IIA 5-brane form an $\mathcal{N}=(2,0)$ tensor multiplet identical to that used in sections 2 and 3. For the compactification on \tilde{T}^5 , four of the five scalars represent the transverse oscillations of the 5-brane in non-compact directions, while one scalar field Y is extracted from one of the RR fields and is indeed periodic. For N IIA 5-branes, as long as they are far apart, we can separate the worldvolume theory into N

different $(2,0)$ tensor multiplets. When they come close together, however, new types of interactions occur. A precise low energy description of this interacting theory is unfortunately not yet available at this point, although some new insights into its structure have been obtained.⁸

Particularly useful information can be extracted by considering the IIA NS 5-branes from the perspective of M-theory⁹. As first pointed out in [26], M2-branes can end on M5-branes via open boundaries in the form of closed strings attached to the M5-brane worldvolume. These membrane boundaries are charged with respect to the self-dual 3-form field strength $T = dU$ on the worldbrane. If we consider an array of N M5-branes, we thus deduce that there are (at least) two basic types of string-like objects on the 5+1-dimensional world volume: strings with variable tension that come from membranes stretched between two different 5-branes, and secondly, strings with fixed tension that correspond to membranes stretched around the 11th direction and returning to the same 5-brane. The first type of strings are charged with respect to (the difference of) the 3-forms of the two 5-branes that are connected by the 2-brane. These charged strings are often referred to as tensionless or self-dual strings. The second type of strings, on the other hand, are neutral with respect to the self-dual 3-form. They represent the micro-strings, *i.e.* the fractionated fundamental IIA strings trapped inside the worldvolume of the N NS fivebranes. The collection of all degrees of freedom together look like an (as yet unknown) non-abelian generalization of the abelian $(2,0)$ theory.

Charge conservation tells us that the charged strings with non-trivial winding numbers around the T^5 are stable against breaking and unwinding. For the neutral strings, on the other hand, there appears to be no similar world-brane argu-

⁸ An (incomplete) list of references include [26][25] [13] [14] [43][44].

⁹ Here in fact we do *not* mean the M(atrrix) theory perspective related via the $T_5 S T_5$ duality chain of section 3, but rather the M-theory whose S_1 compactification directly gives the IIA theory of the last column of Table 1 in section 4.

ment available that would demonstrate their dynamical stability. From the space-time perspective, however, the micro-string winding number is identified with the total NS string winding number (times N), and therefore represents a genuine conserved charge. This conserved quantity is the vector central charge W_i discussed above.

Among the low energy degrees of freedom of the N IIA 5-brane bound state, we can isolate an overall (2,0) tensor multiplet, that describes the collective center of mass motion of the 5-branes. With the help of this decoupled tensor multiplet, we can include the configurations with non-vanishing D-brane charges by turning on the three types of fluxes m_{ij} , p_i and q_5 as described in section 2. The respective IIA and M-theory interpretation of these fluxes are as indicated in table 1 in section 4. In this way we see that Matrix theory indeed exactly reproduces the description in section 2. In addition, we find that the micro-string degrees of freedom, that were first postulated on the basis of U-duality symmetry, have now found a concrete origin.

Based on this new Matrix perspective, we can now obtain a much more detailed description of the M5-brane theory, based on the following proposed Matrix definition of its world-volume theory:

The world-volume theory of the M5-brane at light-cone momentum $p_+ = N/R$ is identical to that of a bound state of N IIA NS 5-branes, subject to ten level matching constraints (14). The KK momenta, $M2$ and $M5$ -brane wrapping numbers are in this correspondence identified with respectively the $D4$, $D2$ and $D0$ -brane charges.

In essence this identification amounts to a definition of the M5-brane as the collection of all point-like bound states of Matrix M-theory on \tilde{T}^5 , including those with zero fivebrane wrapping number. Via this proposed definition, all the properties postulated in [4] have become manifest. The physical motivation and interpretation of the above proposal has been discussed already in section 2. It would clearly be worthwhile to investigate the correspondence between this characterization of the M5-brane world volume theory and

the more conventional low-energy formulation obtained from 11-dimensional supergravity [30].

In the following sections we will to obtain some more detailed dynamical information about the 5+1-dimensional micro-string theory.

8. S_N Orbifolds and 2nd Quantized Strings

Before continuing our journey through the matrix world of M-theory, it will be useful to present a mathematical identity that will be instrumental in establishing a precise link between the present formalism for M-theory and the standard geometric framework of perturbative string theory.

Consider the supersymmetric sigma model on the orbifold target space, defined by the symmetric product space

$$S^N M = M^N / S_N. \quad (42)$$

Here M denotes some target manifold, and S_N denotes the symmetric group, the permutation group of N elements. The two-dimensional world-sheet is taken to be a cylinder parametrized by coordinates (σ, τ) with σ between 0 and 2π . The correspondence we will show is that this sigma model, in the limit $N \rightarrow \infty$, provides an *exact* light-cone gauge description of the Hilbert space of *second quantized* string theory on the target space-time $M \times \mathbf{R}^{1,1}$.

The S_N symmetry is a gauge symmetry of the orbifold model, that permutes all N coordinate fields $X_I \in M$, where $I = 1, \dots, N$ labels the various copies of M in the product M^N . Therefore we can define twisted sectors in which, if we go around the space-like S^1 of the world-sheet, the fields $X_I(\sigma)$ are multivalued, via

$$X_I(\sigma + 2\pi) = X_{g(I)}(\sigma) \quad (43)$$

where g denotes an element of the S_N permutation group. A schematic depiction of these twisted sectors is given in *fig. 2* and correspond to configurations with strings with various lengths.

The Hilbert space of this S_N orbifold field theory is thus decomposed into twisted sectors labeled by the conjugacy classes of the orbifold group S_N [31],

$$\mathcal{H}(S^N M) = \bigoplus_{\text{partitions } \{N_n\}} \mathcal{H}_{\{N_n\}}. \quad (44)$$

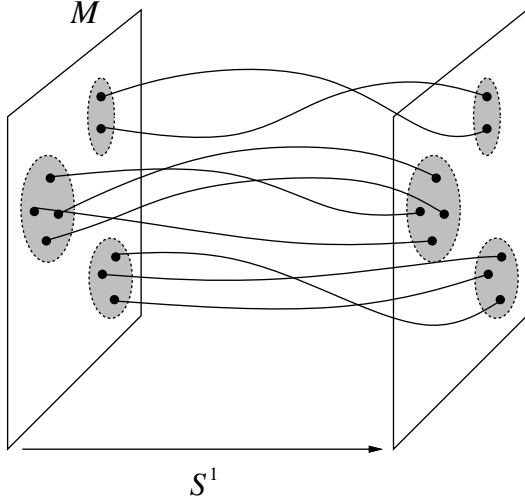


Fig. 2: A twisted sector corresponds to a configuration with strings of various lengths.

Here we used that for the symmetric group, the conjugacy classes $[g]$ are characterized by partitions $\{N_n\}$ of N

$$\sum_n n N_n = N, \quad (45)$$

where N_n denotes the multiplicity of the cyclic permutation (n) of n elements in the decomposition of g

$$[g] = (1)^{N_1} (2)^{N_2} \dots (s)^{N_s}. \quad (46)$$

In each twisted sector, one must further keep only the states invariant under the centralizer subgroup C_g of g , which takes the form

$$C_g = \prod_{n=1}^s S_{N_n} \times \mathbf{Z}_n^{N_n}. \quad (47)$$

Here each factor S_{N_n} permutes the N_n cycles (n) , while each \mathbf{Z}_n acts within one particular cycle (n) .

Corresponding to this factorisation of $[g]$, we can decompose each twisted sector into the product over the subfactors (n) of N_n -fold symmetric tensor products of appropriate smaller Hilbert spaces $\mathcal{H}_{(n)}$

$$\mathcal{H}_{\{N_n\}} = \bigotimes_{n>0} S^{N_n} \mathcal{H}_{(n)} \quad (48)$$

where¹⁰

$$S^N \mathcal{H} = \left(\underbrace{\mathcal{H} \otimes \dots \otimes \mathcal{H}}_{N \text{ times}} \right)^{S_N}. \quad (49)$$

The spaces $\mathcal{H}_{(n)}$ in the above decomposition denote the \mathbf{Z}_n invariant subsector of the space of states of a single string on $\mathbf{R}^8 \times S^1$ with winding number n . We can represent this space via a sigma model of n coordinate fields $X_I(\sigma) \in M$ with the cyclic boundary condition

$$X_I(\sigma + 2\pi) = X_{I+1}(\sigma), \quad (50)$$

for $I \in (1, \dots, n)$. We can glue the n coordinate fields $X(\sigma)$ together into one single field $X(\sigma)$ defined on the interval $0 \leq \sigma \leq 2\pi n$. Hence, relative to the string with winding number one, the oscillators of the long string that generate $\mathcal{H}_{(n)}$ have a fractional $\frac{1}{n}$ moding. The group \mathbf{Z}_n is generated by the cyclic permutation

$$\omega : X_I \rightarrow X_{I+1} \quad (51)$$

which via (50) corresponds to a translation $\sigma \rightarrow \sigma + 2\pi$. Thus the \mathbf{Z}_n -invariant subspace consists of those states for which the fractional left-moving minus right-moving oscillator numbers combined add up to an integer.

It is instructive to describe the implications of this structure for the Virasoro generators $L_0^{(i)}$ of the individual strings. The total L_0 operator of the S_N orbifold CFT in a twisted sector given by cyclic permutations of length n_i decomposes as

$$L_0^{tot} = \sum_i \frac{L_0^{(i)}}{n_i}. \quad (52)$$

Here $L_0^{(i)}$ is the usual canonically normalized operator in terms of the single string coordinates $X(\sigma), \theta(\sigma)$ defined above. The meaning of the above described \mathbf{Z}_n projections is that it requires that the contribution from a single string sector to the total world-sheet translation generator $L_0^{tot} - \bar{L}_0^{tot}$ is integer-valued. From the individual string perspective, this means that $L_0^{(i)} - \bar{L}_0^{(i)}$ is a multiple of n_i .

¹⁰Here the symmetrization is assumed to be compatible with the grading of \mathcal{H} . In particular for pure odd states S^N corresponds to the exterior product \wedge^N .

To recover the Fock space of the second-quantized type IIA string we now consider the following large N limit. We send $N \rightarrow \infty$ and consider twisted sectors that typically consist of a finite number of cycles, with individual lengths n_i that also tend to infinity in proportion with N . The finite ratio n_i/N then represents the fraction of the total p_+ momentum (which we will normalize to $p_+^{tot} = 1$) carried by the corresponding string¹¹

$$p_+^{(i)} = \frac{n_i}{N}. \quad (53)$$

So, in the above terminology, only long strings survive. The usual oscillation states of these strings are generated in the orbifold CFT by creation modes $\alpha_{-k/N}$ with k finite. Therefore, in the large N limit only the very low-energy IR excitations of the S_N -orbifold CFT correspond to string states at finite mass levels.

The \mathbf{Z}_n projection discussed above in this limit effectively amounts to the usual uncompactified level-matching conditions $L_0^{(i)} - \bar{L}_0^{(i)} = 0$ for the individual strings, since all single string states for which $L_0 - \bar{L}_0 \neq 0$ become infinitely massive at large N . The total mass-shell condition reads (here we put the string tension equal to $\alpha' = 1$)

$$p_-^{tot} = N L_0^{tot} \quad (54)$$

The mass-shell conditions of the individual strings is recovered by defining the individual light-cone momenta via the usual relation

$$p_+^{(i)} p_-^{(i)} = L_0^{(i)}. \quad (55)$$

All strings therefore indeed have the same string tension.

In the case that $M = \mathbf{R}^8$, we can choose the fields X^i , θ^a , $\theta^{\dot{a}}$ to transform respectively in the $\mathbf{8}_v$ vector, and $\mathbf{8}_s$ and $\mathbf{8}_c$ spinor representations of the $SO(8)$ R-symmetry group of transversal rotations. The orbifold model then becomes exactly equivalent to the free string limit of the uncompactified type IIA string theory in the Green-Schwarz light-cone formulation. The spacetime

¹¹ The fact that the length n_i of the individual strings specifies its light-cone momentum is familiar from the usual light-cone formulation of string theory.

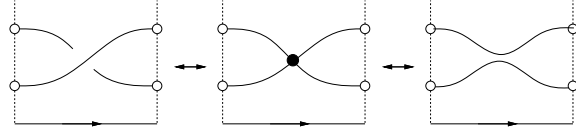


Fig. 3: The splitting and joining of strings via a simple permutation.

supercharges are given by (in the normalization $p_+^{tot} = 1$)

$$Q^a = \frac{1}{\sqrt{N}} \oint d\sigma \sum_{I=1}^N \theta_I^a,$$

$$Q^{\dot{a}} = \sqrt{N} \oint d\sigma \sum_{I=1}^N G_I^{\dot{a}},$$

with

$$G_I^{\dot{a}}(z) = \gamma_{a\dot{a}}^i \theta_I^a \partial x_I^i \quad (56)$$

In the S_N orbifold theory, the twisted sectors of the orbifold corresponding to the possible multi-string states are all superselection sectors. We can introduce string interactions by means of a local perturbation of the orbifold CFT, that generates the elementary joining and splitting of strings.

9. String Interactions via Twist Fields

Consider the operation that connects two different sectors labeled by S_N group elements that are related by multiplication by a simple transposition. It is relatively easy to see that this simple transposition connects a state with, say, one string represented by a cycle (n) decays into a state with two strings represented by a permutation that is a product of two cycles $(n_1)(n_2)$ with $n_1 + n_2 = n$, or vice versa. So the numbers of incoming and outgoing strings differ by one. Pictorially, what takes place is that the two coinciding coordinates, say X_1 and $X_2 \in M$, connect or disconnect at the intersection point, and as illustrated in *fig. 3*. this indeed represents an elementary string interaction.

In the CFT this interaction is represented by a local operator, and thus according to this phys-

ical picture one may view the interacting string theory as obtained via a perturbation of the S_N -orbifold conformal field theory. In first order, this perturbation is described via a modification of the CFT action

$$S = S_{CFT} + \lambda \int d^2z V_{int} \quad (57)$$

where V_{int} is an appropriate twist operator, that generates the just described simple transposition of string coordinates.

Let us consider this operator in more detail for $M = \mathbf{R}^8$. It is clear from the above discussion that the interaction vertex V_{int} will be a twist field that interchanges two eigenvalues, say X_1 and X_2 . It acts therefore as a \mathbf{Z}_2 reflection on the relative coordinate $X_- = X_1 - X_2$, which has 8 components. So we are essentially dealing with a standard $\mathbf{R}^8/\mathbf{Z}_2$ supersymmetric orbifold. This orbifold has well-known twist fields, which will receive contributions of the bosons and fermions. The bosonic twist operator σ is defined via the operator product

$$\partial X_-^i(z) \sigma(0) \sim z^{-\frac{1}{2}} \tau^i(0) \quad (58)$$

Since the fermions transform in spinor representation $\mathbf{8}_s$, their spin fields $\Sigma^i, \Sigma^{\dot{a}}$ will transform in the vector representation $\mathbf{8}_v$ and the conjugated spinor representation $\mathbf{8}_c$. They are related via the operator products

$$\theta_-^a(z) \Sigma^i(0) \sim z^{-\frac{1}{2}} \gamma_{a\dot{a}}^i \Sigma^{\dot{a}}(0) \quad (59)$$

The bosonic twist field σ and the spin fields Σ^i or $\Sigma^{\dot{a}}$ have all conformal dimension $h = \frac{1}{2}$ (i.e. $\frac{1}{16}$ for each coordinate), and the conjugated field τ^i therefore has dimension $h = 1$. For the interaction vertex we propose the following $SO(8)$ invariant operator

$$\tau^i \Sigma^i \quad (60)$$

This twist field has weight $3/2$ and acts within the Ramond sector. It can be written as the $G_{-\frac{1}{2}}$ descendent of the chiral primary field $\sigma \Sigma^{\dot{a}}$, and furthermore satisfies

$$[G_{-\frac{1}{2}}, \tau^i \Sigma^i] = \partial_z(\sigma \Sigma^{\dot{a}}). \quad (61)$$

To obtain the complete form of the effective world-sheet interaction term we have to tensor

the left-moving and right-moving twist fields and to sum over the pairs of I, J labeling the two possible eigenvalues that can be permuted by the \mathbf{Z}_2 twist

$$V_{int} = \sum_{I < J} \left(\tau^i \Sigma^i \otimes \bar{\tau}^j \bar{\Sigma}^j \right)_{IJ}. \quad (62)$$

This is a weight $(\frac{3}{2}, \frac{3}{2})$ conformal field. The corresponding coupling constant λ has therefore total dimension -1 and the interaction will scale linear in g_s just as needed. The interaction preserves the world-sheet $\mathcal{N} = (8, 8)$ supersymmetry, as well as the broken space-time supersymmetries Q^a .

It is interesting to compare the above twist field interaction with the conventional formalism of light-cone string theory. In comparison with the Riemann surfaces picture of interacting strings [32], one should note that around the interaction point the usual string world-sheet is actually a double cover of the local coordinate z , while in the NSR language the above operator insertion also represents a picture changing operation.¹²

We expect that for more general transverse target spaces M the interaction vertex can be constructed quite analogously. Its conformal scale dimension depends on the dimension d of M , or equivalently, the central charge of the corresponding $\mathcal{N}=1$ superconformal field theory, via

$$h = \frac{(d+4)}{8}. \quad (63)$$

In particular we see that the interaction vertex is irrelevant only as long as $d > 4$, and becomes marginal at $d = 4$.

10. Matrix String Theory

Let us now return to the maximally supersymmetric $U(N)$ Yang-Mills theory (31). For generality, let us consider this model in $d+2$ -dimensions on a space-time manifold of the form

$$M_{YM} = T^d \times S^1 \times \mathbf{R} \quad (64)$$

¹²It should also be noted that a similar formulation of light-cone gauge bosonic string theory can be given in terms of an $S^{\tilde{N}} R^{24}$ orbifold. In this case the twist field also carries conformal weight $(3/2, 3/2)$ [33]. In the recent paper [34] it was shown that this formulation indeed exactly reproduces the Virasoro four-point amplitude. A very similar formulation of string perturbation theory was proposed among others by V.Knizhnik [37].

where \mathbf{R} represents time. Strictly speaking, this SYM theory is well-defined only in dimensions $d+2 \leq 4$. In the following, however, we would like to consider the model in a particular IR limit in the S^1 direction, in which the S^1 is taken much larger than the transverse torus T^d . In addition, we wish to combine this limit with the large N limit, similar to the one discussed in the previous sections. Indeed we would like to claim that the dynamics of the maximally supersymmetric YM theory in this limit exactly reduces to that of a (weakly) interacting type IIA string theory on the target space time

$$M_S = T^d \times \mathbf{R}^{10-d}. \quad (65)$$

This particular IR limit effectively amounts to a dimensional reduction, in which the dynamics of the $d+2$ dimensional SYM theory becomes essentially two-dimensional. It is natural and convenient, therefore, to cast the $d+2$ SYM model in the form of a two-dimensional SYM theory. This can be achieved by replacing all fields by infinite dimensional matrices that include the Fourier modes in the transversal directions. In particular, the d transverse components A^i of the gauge potential in the original SYM model thus become d additional two-dimensional scalar fields X^i with $i = 1, \dots, d$ that represent the infinite set of matrix elements in a Fourier basis of the corresponding transverse covariant YM-derivatives. The infinite dimensional matrices X^i defined via this procedure satisfy the recursion relations [38]

$$\begin{aligned} X_{mn}^i &= X_{(m-1)(n-1)}^i, & i > d \\ X_{mn}^i &= X_{(m-1)(n-1)}^i, & i \leq d, m \neq n \\ X_{nn}^i &= 2\pi L + X_{(n-1)(n-1)}^i & i \leq d \end{aligned}$$

With this definition, the $d+2$ dimensional SYM action can be reduced to the following (quasi) two-dimensional form

$$\begin{aligned} S_{2D} &= \int \text{tr} \left((D_\alpha X^I)^2 + \psi \Gamma^\alpha D_\alpha \psi + \lambda^2 F_{\alpha\beta}^2 \right. \\ &\quad \left. + \frac{1}{\lambda} \psi \Gamma^I [X^I, \psi] + \frac{1}{2\lambda^2} [X^I, X^J]^2 \right) \end{aligned}$$

The interpretation of this $\mathcal{N} = 8$ SYM model as a matrix string theory is based on the identification

of the set of eigenvalues of the matrix coordinates X^i with the coordinates of the fundamental type IIA strings. In addition, the coupling λ is given in terms of the string coupling g_s as

$$\lambda^2 = \alpha' g_s^2. \quad (66)$$

The dependence on the string coupling constant λ can be absorbed in the area dependence of the two-dimensional SYM model. In this way λ scales inversely with world-sheet length. The free string at $\lambda = 0$ is recovered in the IR limit. In this IR limit, the two-dimensional gauge theory model is strongly coupled and we expect a nontrivial conformal field theory to describe the IR fixed point.

It turns out that we can find this CFT description via the following rather naive reasoning. We first notice that in the $g_s = 0$ limit, the last two potential terms in the 2-dimensional action (66) effectively turn into constraints, requiring the matrix fields X and θ to commute. This means that we can write the matrix coordinates X^i in a simultaneously diagonalized form

$$X^i = U \text{diag}(x_1^i, \dots, x_N^i) U^{-1} \quad (67)$$

with $U \in U(N)$. Here U and all eigenvalues x_I^i can of course still depend on the world-sheet coordinates.

Once we restrict ourselves to Higgs field configurations of the form (67), the two-dimensional gauge field A_α decouples. To make this explicit, we can use the YM gauge invariance to put $U = 1$ everywhere, which in particular implies that the YM-current $j_{YM} = [X^i, \partial_\alpha X_i]$ vanishes. The only non-trivial effect that remains is that via the Higgs mechanism, the charged components of A_α acquire a mass for as long as the eigenvalues x_i are distinct and thus decouple in the infra-red limit. This leads to a description of the infra-red SYM model in terms of N Green-Schwarz light-cone coordinates formed by the eigenvalues $x_I^i, \theta_I^a, \theta_I^{\dot{a}}$ with $I = 1, \dots, N$.

The $U(N)$ local gauge invariance implies that there is still an infinite set of discrete gauge transformations that act on the diagonalized configurations (67). Part of these discrete gauge transformations act via translations on the eigenvalues x_I^i that represent the transversal components of

the gauge potential, turning them into periodic variables. In addition, we must divide out by the action of the Weyl group elements of $U(N)$, that act on the eigenvalues x_I^i via S_N permutations. Thus, via this semi-classical reasoning, we deduce that the IR conformal field theory is the supersymmetric sigma model on the orbifold target space

$$(T^d \times \mathbf{R}^{8-d})^N / S_N. \quad (68)$$

When we combine this with the results described in the previous two sections, we deduce that the super-Yang-Mills model in the strict IR limit gives the same Hilbert space as the free light-cone quantization of second quantized type IIA string theory.

The conservation of string number and individual string momenta will be violated if we turn on the interactions of the 1+1-dimensional SYM theory. When we relax the strict IR limit, one gradually needs to include configurations in which the non-abelian symmetry gets restored in some small space-time region: if at some point in the (σ, τ) plane two eigenvalues x_I and x_J coincide, we enter a phase where an unbroken $U(2)$ symmetry is restored. We should thus expect that for non-zero λ , there will be a non-zero transition amplitude between states that are related by a simple transposition of these two eigenvalues.

We now claim that this joining and splitting process is indeed first order in the coupling constant λ as defined in the SYM Lagrangian. Instead of deriving this directly in the strongly coupled SYM theory, we can analyze the effective operator that produces such an interaction in the IR conformal field theory. The identification $\lambda \sim g_s \sqrt{\alpha'}$ requires that this local interaction vertex V_{int} must have scale dimension

$$(h, \bar{h}) = \left(\frac{3}{2}, \frac{3}{2}\right) \quad (69)$$

under left- and right-scale transformations on the two-dimensional world-sheet. The analysis of the previous section confirms that this is indeed the case.

As a further confirmation of this physical picture, it is relevant to point out that there is no classical barrier against the occurrence of string

interactions in the IR limit. For every string world-sheet topology, there exists a solution to the SYM equation of motion of the form (67) such that the eigenvalues x_I^i form this string world-sheet via an appropriate branched covering of the cylinder $S^1 \times \mathbf{R}$. (An explicit description of these solutions is given in [35]). The suppression of string interactions in the IR field theory therefore entirely arises due to the quantum fluctuations around these classical field configurations. This suggests that there should exist a close relation between the moduli space of classical configurations of the two-dimensional SYM model in the IR regime and the universal moduli space of all two-dimensional Riemann surfaces.

From this Yang-Mills set-up, it is straightforward to derive the correct scaling dimension of the twist-fields (that create the branch cuts) by computing the one-loop fluctuation determinant around such a non-trivial classical field configuration. To start with, this computation only involves the eigenvalue fields x_I^i , and thereby reduces to the standard analysis of twist field operators in orbifold CFT used above. In the present context, however, the orbifold CFT is automatically provided with an ultra-violet cut-off of the order λ (as defined in (66)), since this is the place where the non-abelian SYM dynamics takes over. This explains why the coupling λ in (66) also determines the strength of the twist field perturbation away from the orbifold CFT.

It would clearly be interesting to make via this set-up a direct quantitative comparison with the standard perturbative SYM calculations. In [7][9] [36] it was shown that the SYM amplitudes at weak coupling reproduce the tree level graviton exchange amplitude of eleven dimensional supergravity. Non-renormalization theorems of the SYM-model should ensure that these specific amplitudes receive no higher order or non-perturbative corrections, so that we can extrapolate these results the IR limit. Via the comparison with our IR description of the same amplitudes, one can thus uniquely fix the ratio between the orbifold perturbation parameter (*i.e.* the string coupling) and the coupling constant λ in the two-dimensional SYM lagrangian 66.

The result (69) for the conformal dimension of the interaction vertex V_{int} of course crucially depends on the number of Higgs scalar fields being equal to 8. The fact that for this case V_{int} is indeed irrelevant in the IR limit provides the justification afterwards of the above naive derivation of the orbifold sigma model as describing the IR phase of the $U(N)$ SYM model. Another way of stating the same result is that the $\mathbf{R}^8/\mathbf{Z}_2$ cannot be resolved to give a smooth Calabi-Yau space.

It is indeed interesting to try to repeat the same study for SYM models with less supercharges. Concretely, we could consider the dimensional reduction of the $\mathcal{N} = 1$ SYM model in 5+1-dimensions, which will give us a matrix string theory with $d = 4$ transverse Higgs fields. Via the above reasoning we then deduce that in the strict IR limit in the S^1 direction, the string interactions are not turned off, since the interaction vertex V_{int} is now exactly marginal. We will return to this case in the next section.

Finally we remark that, for SYM theories with even less supersymmetries, the naive reasoning applied in this section in fact breaks down, since in this case the interaction vertex V_{int} becomes a relevant perturbation. Hence in this case the correct IR dynamics is no longer accurately represented by a (small) deformation of the naive orbifold IR field theory. Instead one expects that the twist field V_{int} obtains a non-zero expectation value, and the IR theory presumably takes the form of some Ising type QFT.

11. Microstring Interactions

A relatively concrete model of the micro-string theory can be obtained by applying the reasoning of [14] to the 5+1 SYM formulation of Matrix theory. Namely, in this formulation we can describe longitudinal NS 5-branes (that extend in the light-cone direction) by means of YM instanton configurations [39]. Here the 5-brane number simply coincides with the instanton number k .¹³ Similar as in [14], we can then consider the

¹³This representation follows from the table in section 7, where the longitudinal M5 branes (via their IIA identification with D4-branes) get mapped by the T^5 duality to D1 strings, which manifests themselves as instantons in

world-volume theory of these k longitudinal NS 5-branes in the limit of zero string coupling. As also explained above, this corresponds to taking the limit where one of the compactification circles becomes very large. The theory thus again reduces to the IR limit a two-dimensional matrix string theory of the form (66), but now with matrices X^I that carry a non-zero topological charge [39][20]

$$k = \text{tr} X_{[i} X_j X_k X_{l]} \quad (70)$$

representing the instanton number on the transverse T^4 . It can be seen that in this case, the IR limit of the matrix string theory (66) no longer becomes a free field theory but instead reduces to a sigma model on the moduli space $\mathcal{M}_{N,k}(\hat{T}^4)$ of k $U(N)$ instantons on the four-torus \hat{T}^4 . Since this instanton moduli space is hyperkähler, the corresponding supersymmetric sigma model is indeed conformally invariant.

In [13] it was proposed that this sigma-model could be used as a concrete way of representing the interacting micro-string theory on the world-volume of the k 5-branes. Crucial for this interpretation is the fact that the moduli space of instanton configurations on T^4 with rank N , instanton number k , and fluxes m can be thought of as a (hyperkähler) deformation of the symmetric product space $S^n T^4$ with $n = Nk + \frac{1}{2}m \wedge m$. Hence via the results of the previous sections, the sigma model on $\mathcal{M}_{N,k}(\hat{T}^4)$ is indeed expected to behave in many respects as a light-cone description of a 5+1-dimensional interacting string theory.

The interpretation of this sigma model as describing the micro-string dynamics on k five-branes becomes a bit more manifest via the so-called Mukai-Nahm duality transformation. This transformation provides an isometry

$$\mathcal{M}_{N,k}(\hat{T}^4) = \mathcal{M}_{k,N}(T^4) \quad (71)$$

between the moduli space of k $U(N)$ instantons on \hat{T}^4 to the moduli space of N instantons on a $U(k)$ Yang-Mills theory on the dual four-torus T^4 . This dual $U(k)$ SYM theory can be thought of as the world-volume theory of k fivebranes wrapped the large N D5 SYM theory.

around T^4 , and the N instantons describe the micro-string degrees of freedom on this world-volume. This dual language is most suitable for discussing the decompactification limit of the M-theory torus, which leads to the description of k uncompactified fivebranes as a non-linear sigma model on $\mathcal{M}_{k,N}(\mathbf{R}^4)$.

The fact that there exists a superconformal deformation of the orbifold sigma model on $S^n T^4$ is in direct accordance with the results of the previous section, since it is reasonable to suspect that the marginal operator that represents this deformation can be identified with an appropriate \mathbf{Z}_2 twist operator that generates the splitting and joining interactions among the microstrings. As the string in this case has 4 transverse dimensions, the twist-field indeed has total dimension 2. Below we would like to make this description more explicit.

To this end, it will be useful to consider the situation in Matrix theory, which describes the k 5-branes in combination with a single IIA matrix string. In fact, to simplify the following discussion, we will give this string the minimal p_+ momentum, that is we will represent it simply by adding one row to the matrix coordinates X^i , so that we are dealing with a $U(N+1)$ matrix theory. Next we write

$$X_i \rightarrow \begin{pmatrix} Z_i & Y_i \\ \bar{Y}_i & x_i \end{pmatrix} \quad (72)$$

Here the matrix entry x_i represents the coordinate of a propagating type IIA string (with infinitesimal light-cone momentum p_+), and the $U(N)$ part Z_i describes the k 5-branes. Similarly, we can write

$$\theta_a \rightarrow \begin{pmatrix} \zeta_a & \eta_a \\ \bar{\eta}_a & \theta_a \end{pmatrix} \quad (73)$$

The idea is now to fix Z_i to be some given instanton configuration on T^4 plus a diagonal part $z_I \mathbf{1}_{N \times N}$ that describes the location of the 5-brane in the uncompactified space, and to consider the propagation of the IIA string in its background.

This set-up will be useful in fact for studying two separate regimes, each of which are quite interesting [44] [45]. First there is the micro-string regime, which corresponds to the case where the

extra string x is captured inside the k NS 5-branes. In this case, the Y variable need to be thought of as the extra degrees of freedom that extend the moduli space of k $U(N)$ instantons to that of k $U(N+1)$ instantons. Secondly, we can also consider the regime where the IIA string described by the x -coordinate is far enough removed from the NS 5-branes so that it propagates just freely on the background geometry of the 5-brane solitons. This outside regime corresponds to the Coulomb branch of the matrix string gauge-theory, while the micro-string regime represents the Higgs branch.

For both regimes, it will in fact be useful to isolate from the Y degrees of freedom in (72) the part that describes the constant zero-modes in the instanton background, satisfying

$$Z_{[i} Y_{j]} = 0, \quad \gamma_{\alpha\beta}^i Z_i \eta^\beta = 0. \quad (74)$$

Since Z_i corresponds to a k -instanton configuration we can use the index theorem to determine that there are k such zero-modes, both for the Y and η field. In the Coulomb phase they describe the low energy degrees of freedom that mediate the interaction between the string and the 5-brane. In this phase, the Y -modes are massive and thus are naturally integrated out in taking the IR limit on the S^1 . In the Higgs phase, on the other hand, they become massless two-dimensional fields, and represent the additional micro-string degrees of freedom that arise from the IIA string trapped inside the k 5-branes. Notice that in this transition from the Coulomb phase to the Higgs phase the single IIA string indeed “fractionates” into k individual micro-strings.

The quadratic part of the action for the Y degrees of freedom takes the form

$$\begin{aligned} S = & \int d^2\sigma \left((DY_i)^2 + |x_I - z_I|^2 |Y_i|^2 \right) \\ & + \int d^2\sigma \left(\bar{\eta}_a^\alpha D_+ \eta_a^\alpha + \bar{\eta}_a^{\dot{\alpha}} D_- \eta_a^{\dot{\alpha}} \right. \\ & \left. + \bar{\eta}_a^\alpha \gamma_{\alpha\dot{\beta}}^I (x - z)_I \eta_a^{\dot{\beta}} \right) \end{aligned} \quad (75)$$

Here we suppressed the summation over the index labeling the k zero-modes. The discrete symme-

tries of the instanton moduli space imply that the model (75) possesses a S_k gauge symmetry that acts by permuting these k coordinates. In [42] the presence of the fivebrane background was incorporated in matrix theory by adding an appropriate hypermultiplet to the Matrix action. In our set-up, the role of these additional hypermultiplets is taken over by the off-diagonal fields Y, η .

Let us explain the fermion chiralities as they appear in the Lagrangian (75). In the original matrix string action the $SO(8)$ space-time chiralities are directly correlated with the world-sheet chirality, reflecting the type IIA string setup, giving left-moving fermions θ^a and right-moving fermions $\theta^{\dot{a}}$. The T^4 compactification breaks the $SO(8)$ into an internal $SO(4)$ that acts inside the four-torus times the transversal $SO(4)$ rotations in space-time. The spinors decompose correspondingly as $\mathbf{8}_s \rightarrow (\mathbf{2}_+, \mathbf{2}_+) \oplus (\mathbf{2}_-, \mathbf{2}_-)$ and $\mathbf{8}_c \rightarrow (\mathbf{2}_+, \mathbf{2}_-) \oplus (\mathbf{2}_-, \mathbf{2}_+)$. The instanton on T^4 with positive charge $k > 0$ allows only zero modes of positive chirality, selecting the spinors that transform as $\mathbf{2}_+$ of the first $SO(4)$ factor. So, through this process the space-time chirality of the fermion zero modes gets correlated with the world-sheet handedness, giving k left-moving fields η_a^α and k right-moving fields $\eta_a^{\dot{\alpha}}$, where $\alpha, \dot{\alpha} = 1, 2$ denote chiral space-time spinor indices and $a = 1, 2$ label chiral world-volume spinors.

The Higgs phase of the matrix string gauge theory corresponds to the regime where $x_I - z_I \simeq 0$. The above lagrangian of the $4k$ string-coordinates (Y, η) then represents a linearized description of the micro-string motion. inside the fivebrane worldvolume. The linearized world-sheet theory has 4 left-moving and 4 right-moving supercharges

$$F^{\dot{a}\alpha} = \oint \partial Y^{\dot{a}b} \eta_b^\alpha, \quad \bar{F}^{\dot{a}\dot{\alpha}} = \oint \bar{\partial} Y^{\dot{a}b} \bar{\eta}_b^{\dot{\alpha}} \quad (76)$$

which correspond to the unbroken part of the world-brane supersymmetry and satisfy

$$\{F^{\dot{a}\alpha}, F^{\dot{b}\beta}\} = 2\epsilon^{\dot{a}\dot{b}}\epsilon^{\alpha\beta} L_0, \quad (77)$$

Here L_0 is the left-moving world-sheet Hamiltonian.

The ground states of the micro-strings contain the low-energy collective modes of the five-

brane. To make this explicit, we note that the left-moving ground state must form a multiplet of the left-moving fermionic zero-mode algebra

$$\{\eta_a^\alpha, \eta_b^\beta\} = \epsilon_{ab}\epsilon^{\alpha\beta} \quad (78)$$

which gives 2 left-moving bosonic ground states $|\alpha\rangle$ and 2 fermionic states $|a\rangle$. By taking the tensor product with the right-moving vacua one obtains in total 16 ground states, which represent the massless tensor multiplet on the five-brane. Specifically, we have states $|\alpha\rangle \otimes |\dot{\beta}\rangle$ that describe four scalars $X^{\alpha\dot{\beta}} = \sigma_i^{\alpha\dot{\beta}} X^i$ transforming a vector of the $SO(4)$ R-symmetry, while $|\alpha\rangle \otimes |a\rangle$ and $|a\rangle \otimes |\dot{\alpha}\rangle$ describe the fermions ψ_a^α and $\psi_a^{\dot{\alpha}}$. Finally, the RR-like states $|ab\rangle$ decompose into the fifth scalar Y , and the 3 helicity states of the self-dual tensor field. These are indeed the fields that parametrize the collective excitations of the NS-brane soliton¹⁴.

This 5+1-dimensional string thus forms a rather direct generalization of the 9+1-dimensional Green-Schwarz string. Clearly, however, the linearized theory can not be the whole story, because it would not result in a Lorentz covariant description of the M5 world-brane dynamics. Micro string theory is indeed essentially an interacting theory, without a weak coupling limit. Its interacting dynamics is obtained by including the non-linear corrections to (75), obtained by properly taking into account the non-linear geometry of the moduli space $\mathcal{M}_{N,k}(\hat{T}^4)$ of $U(N)$ k -instantons on the four-torus \hat{T}^4 .

As argued above, the sigma model on $\mathcal{M}_{N,k}(\hat{T}^4)$ should be representable in terms of a finite marginal deformation of the above model by means of a twist field. Concretely, the linearized orbifold model has four different twist fields

$$V_{ab} = V_a \otimes \bar{V}_b \quad (79)$$

where the left-moving factor V_a is

$$V_a = \gamma_{ab}^i \tau_i \Sigma^b \quad (80)$$

and a similar expression holds for the right-moving \bar{V}_a . Here Σ^b denotes the fermionic spin field, defined via the operator product relation

$$\eta_a^\alpha(z) \Sigma_b(0) \sim z^{-\frac{1}{2}} \epsilon_{ab} \tilde{\Sigma}^\alpha(0) \quad (81)$$

¹⁴A similar CFT description of the fivebrane effective field theory has been proposed in [40].

with $\tilde{\Sigma}^\alpha(0)$ the dual spin field. Both spin fields have conformal dimension $1/4$, while τ_i has dimension $3/4$.

So, all the above operators describe possible marginal deformations, that at least in first order conformal perturbation theory, preserve the conformal invariance of the sigma model QFT. These 4 physical operators V_{ab} are in one-to-one correspondence with the RR-like micro-string ground states $|a\rangle \otimes |b\rangle$, that represent the excitations of $T = dU$ and the scalar field Y . Rotation symmetry on the world-volume, however, selects out the scalar combination

$$V_Y = \epsilon^{ab} V_a \otimes \bar{V}_b \quad (82)$$

that describes the emission of a Y mode, as the only good candidate for representing the string interactions.

Geometrically, the twist fields V_{ab} represent the four independent blow-up modes of a $\mathbf{R}^4/\mathbf{Z}_2$ orbifold singularity. The three symmetric combinations deform the orbifold metric to a smooth ALE space and the three parameters correspond to the different complex structures that one can pick to specify the blow-up. These deformations lead to regular supersymmetric sigma models, but are not rotational invariant. The antisymmetric combination V_Y can be interpreted as turning on a B -field θ -angle dual to the vanishing two-cycle around the \mathbf{Z}_2 orbifold point. The value of this θ -angle away from the pure orbifold CFT, which determines the strength of the micro-string interactions, is in principle fixed by the description of the matrix micro-string as the sigma-model on $\mathcal{M}_{N,k}(\hat{T}^4)$.

The value of the θ -angle modulus can be determined by considering the instanton moduli space around the locus of point-like instantons. This problem was considered in [44], where it is proposed that this naturally fixes its value at $\theta = \pi$.¹⁵ A puzzling aspect of this proposal is that this special value of θ is actually known to lead to a singular CFT [46]. This is problematic, since it would make the sigma model description of the micro-string dynamics inconsistent, or at

least incomplete. There could however be a good physical reason for the occurrence of this singular CFT. In the linear sigma model description of the Higgs branch, the singularity seems closely related to the presence of a Coulomb branch on the moduli space that describes the emission of strings from the fivebrane [44]. An alternative physical interpretation, however, is that the singularity of the CFT reflects the possible presence of the tensionless charged strings.

The latter interpretation in fact suggests that the singularity may possibly be resolved by introducing a finite separation between the k fivebranes. In the M-theory decompactification limit in particular, we know that the relevant moduli space $\mathcal{M}_{k,N}(\mathbf{R}^4)$ allows for a number of deformation parameters in the form of the Higgs expectation values of the $U(k)$ SYM model. These Higgs parameters modify the geometry of the instanton moduli space, and indeed have the effect of softening most of its singularities. This is in direct accordance with the micro-string perspective, since for finite Higgs expectation values the charged self-dual strings always have a finite tension.

12. Future Directions

To end with, we would like to mention some future directions, where useful new insights may be obtained.

Space-time Geometry.

We have seen that the matrix setup is quite well adapted to extract all the effective degrees of freedom that describe the type IIA fivebrane, and one might hope that the present context of matrix string theory opens up new ways of examining the interaction with their environment. In particular, at weak coupling one would like to be able to recover the classical fivebrane geometry including its anti-symmetric tensor field. This geometry should arise in the CFT description of the Coulomb phase of the matrix string theory, where the string $x(\sigma)$ propagates at a finite distance $|x_I - z_I| > 0$ from the fivebrane soliton.

This description of the string and fivebrane is closely related to the recently studied case of a D-string probe in the background of a D-fivebrane

¹⁵Here we defined $\theta = 0$ as the value at the free orbifold CFT.

[41]. One can derive an effective action for the x -part of the $U(N)$ matrix by integrating out the off-diagonal components Y and η . In [41] this was shown that the one-loop result for the transversal background metric

$$ds^2 = \left(1 + \frac{k}{|x - z|^2}\right)(dx)^2. \quad (83)$$

is indeed the well-known geometry of the NS five-brane [47].

Most characteristic of the solitonic fivebrane is its magnetic charge with respect to the NS anti-symmetric tensor field B

$$\int_{S_3} H = k \quad (84)$$

where $H = dB$ for a three sphere enclosing the brane. This result can be recovered analogously, via a direct adaptation of the analysis of [42]. The transversal rotation group $SO(4) \simeq SU(2)_L \times SU(2)_R$ acts on the relative coordinate field $(x - z)^I$ via left and right multiplication. This group action can be used to decompose the 2×2 matrix $\gamma_I^{\alpha\beta}(x - z)^I$ in terms of a radial scalar field $r = e^\varphi$ and group variables g_L, g_R as

$$\gamma_I(x - z)^I = e^\varphi g_L g_R^{-1}. \quad (85)$$

The combination $g = g_L g_R^{-1}$ labels the $SO(3)$ angular coordinate in the transversal 4-plane. Inserting the above expression for x^I into (75), the angles g_L and g_R can be absorbed into η via a chiral rotation, producing a model of k $SU(2)$ fermions coupled to a gauge field $A_+ = g_L^{-1} \partial_+ g_L$, $A_- = g_R^{-1} \partial_- g_R$. Via the standard chiral anomaly argument, one derives that the one-loop effective action includes an $SU(2)$ WZW-model for the field g with level given by the five-brane number k . The background three-form field H in (84) is reproduced as the Wess-Zumino term of this action.

We thus recover the CFT description of the type IIA string moving near the fivebrane [47]. This action consists of a level k $SU(2)$ WZW model combined with a Liouville scalar field φ

$$S(x^I) = S_{WZW}(g) + S_L(\varphi). \quad (86)$$

with

$$S_L(\varphi) = \int d^2\sigma \left(|\partial\varphi|^2 + \gamma R^{(2)}\varphi \right) \quad (87)$$

The background charge γ of the scalar φ represents a space-time dilaton field that grows linearly for $\phi \rightarrow -\infty$, and is normalized such that the total central charge is $c = 6$.

It is quite meaningful that the radial coordinate $r = e^\varphi$ is described in terms of a Liouville type field theory (87). The role of r is indeed very similar to that of the world-sheet metric in two-dimensional quantum gravity: since r represents the mass of the lightest fields that have been integrated out, it also parametrizes the short-distance world-sheet-length scale at which the CFT description is expected to break down. Besides the fact that it acquires a corresponding conformal scale dimension, this also means that it provides the natural cut-off scale at which the short-distance singularities need to be regulated.

In this light, it would be very interesting to study in more detail the transition region between the Coulomb and Higgs phase of this two-dimensional gauge theory, describing the capturing process of the micro-string inside the five-brane. The relevant space-time geometry in this case should contain all the essential features of a black hole horizon, which initially is located at $r = e^\varphi = 0$. We expect that via the above CFT correspondence, and in particular by studying the limits on its validity, useful new insights into the reliability of semi-classical studies of black hole physics can be obtained [48,13].

Micro-strings on $K3 \times T^2$.

Interesting new lessons can be learned from the beautiful geometric realization of heterotic string theory via M-theory compactified on $K3$. In this correspondence the perturbative heterotic string becomes a natural part of the M5-brane spectrum wrapped around the $K3$ [49], which thereby provides a far reaching generalisation of its world-sheet dynamics in terms of the above 5+1-dimensional micro-string theory. A matrix formulation of the interactions between different wrapped M5-branes may ultimately provide a description of heterotic string dynamics, in which non-perturbative duality is realized as a geometric symmetry.

As a concrete illustration, we consider the generalization of the one-loop heterotic string par-

tion function. In perturbative string theory, this one-loop calculation proceeds by considering the CFT free energy on a world-sheet with the topology of T^2 , *i.e.* the closed world-trajectory of a closed string. In the M-theory realization, the string world-sheet expands to become the world-volume of the M5-brane, while the CFT get replaced by the micro-string theory. The one-loop diagram thus takes the form of a M5-brane world-trajectory with the topology of $K3 \times T^2$. The direct analogue of the perturbative string calculation thus involves the partition function second quantized micro-string theory on this 6-dimensional manifold.

For the restricted case where one considers only the BPS contributions, this calculation can be explicitly done. The BPS restriction essentially amounts to a projection on only the left-moving micro-string modes, and thereby eliminates the effect of its interactions. The M5-brane BPS partition sum can thus be computed by taking the exponent of the relevant one-loop free energy of the micro-string

$$\mathcal{Z}_{M5-brane} = \exp \mathcal{F}_{1-loop}^{micro-string} \quad (88)$$

The above one-loop string partition function has in fact been computed already in [51]. Let us briefly describe the result, because it in fact displays some rather suggestive symmetries.

This partition sum \mathcal{Z} will depend on the moduli that parametrize the shape and size of world-volume manifold $K3 \times T^2$. It turns out that there are three relevant moduli T , U and V , that parametrize the shape and size of the T^2 and a Wilson line expectation value. The expression for the partition function (88) now takes the following form¹⁶

$$\mathcal{Z}(T, U, V) = e^{-2\pi i(T+U+V)} \times \prod_{(k,l,m) > 0} \left(1 - e^{2\pi i(kT+lU+mV)}\right)^{-c(4kl-m^2)} \quad (89)$$

where the coefficients $c(n)$ are defined by the ex-

¹⁶ Here $(k, l, m) > 0$ means that $k, l \geq 0$ and $m \in \mathbf{Z}$, $m < 0$ for $k = l = 0$.

pansion of the $K3$ elliptic genus as

$$\chi_{\tau,z}(K3) = \sum_{h \geq 0, m \in \mathbf{Z}} c(4h - m^2) e^{2\pi i(h\tau + mz)} \quad (90)$$

and are non-zero for $n \geq -1$. The expression (89) can indeed be recognized as a second quantized string partition function [31]. If we expand \mathcal{Z} as a formal Fourier series expansion

$$\mathcal{Z}(T, U, V) = \sum_{k,l,m} D(k, l, m) e^{-2\pi i(kT+lU+mV)} \quad (91)$$

the integer coefficients $D(k, l, m)$ represent the total (bosonic minus fermionic) number of states with given total momentum, winding and $U(1)$ charge, as specified by the three integers k, l, m .

The above partition function can indeed be thought as the direct generalization of the genus one partition function of perturbative heterotic string BPS-states. The fact that \mathcal{Z} now depends on three moduli instead of one reflects the fact that the worldsheet CFT is now replaced by a world-volume micro-string theory, which in particular also has winding sectors that are sensitive to the size of the T^2 . As a consequence, the partition function \mathcal{Z} has the intriguing property that it is symmetric under duality transformations, that in particular interchange the shape and size parameters T and V .

The general group of dualities is best exhibited by writing the three moduli into a 2 by 2 matrix

$$\Omega = \begin{pmatrix} T & V \\ V & U \end{pmatrix} \quad (92)$$

which takes the suggestive form of a period matrix of a genus 2 curve. The above partition function \mathcal{Z} indeed turns out to be a modular form under the corresponding $SO(3, 2; \mathbf{Z})$ modular group, and is in fact *invariant* under a $SL(2, \mathbf{Z})$ subgroup. It is quite tempting to identify this $SL(2, \mathbf{Z})$ symmetry with the electro-magnetic duality symmetry that arises upon further compactification of this theory to four dimensions. A concrete suggestion that was made in [50] is that the coefficients in the expansion (91) represent the degeneracy of (bosonic minus fermionic) dyonic BPS states with a given electric charge $q_e \in \Gamma^{22,4}$

and magnetic charge $q_m \in \Gamma^{22,4}$, via the identification

$$d(q_m, q_e) = D(\frac{1}{2}q_m^2, \frac{1}{2}q_e^2, q_e \cdot q_m) \quad (93)$$

In [50] it was shown that this formula survives a number of non-trivial tests, such as correspondence with the perturbative heterotic string as well as with the Bekenstein-Hawking black hole entropy formula. If true, this result suggests that the presented formulation of M-theory via the microstring theory and the M5-brane may in fact be extended to compactifications all the way down to four dimensions. Clearly, further study of matrix theory compactifications is needed to confirm or disprove this conjecture.

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